EQUILIBRIUM STATES OF INTERVAL MAPS FOR HYPERBOLIC POTENTIALS

HUAIBIN LI AND JUAN RIVERA-LETELIER

Abstract. We study the thermodynamic formalism of sufficiently regular interval maps for Hölder continuous potentials. Extending a previous result of Bruin and Todd, we show that if the potential is hyperbolic and satisfies a technical bounded distortion hypothesis, then there is a unique equilibrium state, and that this measure is exponentially mixing. Moreover, we show the absence of phase transitions: The pressure function is real analytic at such a potential.

1. Introduction

In this paper we study the thermodynamic formalism of sufficiently regular interval maps for Hölder continuous potentials. The case of a piecewise monotone interval map $f : I \to I$, and a potential φ of bounded variation satisfying

$$\sup_I \varphi < P(f, \varphi),$$

where $P(f, \varphi)$ denotes the pressure, is very well understood. Most results apply under the following weaker condition:

For some integer $n \geq 1$, the function $S_n(\varphi) := \sum_{j=0}^{n-1} \varphi \circ f^j$ satisfies $\sup_I \frac{1}{n} S_n(\varphi) < P(f, \varphi)$.

In what follows, a potential φ satisfying this condition is said to be *hyperbolic for f*. See for example [BK90, DKU90, HK82, Kel85, LSV98, Rue94] and references therein, as well as Baladi's book [Bal00, §3]. The classical result of Lasota and Yorke [LY73] corresponds to the special case where f is piecewise C^2 and uniformly expanding, and $\varphi = -\ln |Df|$.

For a complex rational map in one variable f, and a Hölder continuous potential φ that is hyperbolic for f, a complete description of the thermodynamic formalism was given by Denker, Haydn, Przytycki, and Urbański in [Hay99, DPU96, DU91, Prz90],* extending previous results of Freire,

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*In this setting, most of the results have been stated for a potential φ satisfying the condition $\sup \varphi < P(f, \varphi)$ that is more restrictive than φ being hyperbolic for f. General arguments show they also apply to hyperbolic potentials, see [IRRL12, §3].

Lopes, and Mañé [FLM83, Mañ83], and Ljubich [Lju83]. See also the alternative approach of Szostakiewicz, Urbański, and Zdunik in [SUZ11].

In both of these settings, piecewise monotone maps for bounded variation potentials, and complex rational maps for Hölder continuous potentials, most of the arguments rely on a study of the corresponding transfer operator. So the tools developed to prove these results breakdown in the case of a sufficiently regular interval map with a Hölder continuous potential, since in most cases the space of Hölder continuous potentials is not invariant by the corresponding transfer operator.†

Using an inducing scheme, Bruin and Todd obtained a general result for potentials that satisfy the "bounded range" condition:

$$\sup_I \varphi - \inf_I \varphi < h_{\mathrm{top}}(f), \tag{1.1}$$

where $h_{\mathrm{top}}(f)$ denotes the topological entropy of f, see [BT08, Theorem 4] and also [IT11, Theorem 1.1]. This condition is more restrictive than hyperbolicity, see Appendix A.

The purpose of this paper is to extend [BT08, Theorem 4] to Hölder continuous potentials that are hyperbolic. Our result holds under a technical bounded distortion hypothesis, similar to that of [BT08, Theorem 4]. This hypothesis is automatically satisfied if the map is weakly hyperbolic in a precise sense. The fact that our results hold for hyperbolic potentials, and not only for the more restricted class of potentials satisfying the bounded range condition, is crucial to obtain the main results of the companion paper [LRL12]. To prove our main result, we use the induced scheme developed by Przytycki and the second named author in [PRL07, PRL11].

We now proceed to state our main result more precisely.

1.1. **Statement of results.** Let (X, dist) be a compact metric space and $T : X \to X$ a continuous map. Denote by $\mathcal{M}(X)$ the space of Borel probability measures on X endowed with the weak* topology, and by $\mathcal{M}(X,T)$ the subspace of $\mathcal{M}(X)$ of those measures that are invariant by T. For each measure ν in $\mathcal{M}(X,T)$, denote by $h_\nu(T)$ the *measure-theoretic entropy* of ν. For a continuous function $\varphi : X \to \mathbb{R}$, denote by $P(T,\varphi)$ the *topological pressure of T for the potential φ*, defined by

$$P(T,\varphi) := \sup\left\{ h_\nu(T) + \int \varphi \, d\nu : \nu \in \mathcal{M}(X,T) \right\}. \tag{1.2}$$

A measure ν in $\mathcal{M}(X,T)$ is called an *equilibrium state of T for the potential φ*, if the supremum in (1.2) is attained at ν.

On the other hand, a measure ν in $\mathcal{M}(X,T)$ is *exponentially mixing for f*, or *has exponential decay of correlations for f*, if there are constants $C > 0$

†The spaces of functions of p-bounded variation used by Keller in [Kel85], which contain Hölder continuous functions as well as functions with discontinuities, are well adapted for interval maps of Lorenz type. However, these spaces are not invariant by the transfer operator of a map having a non-flat critical point.

and ρ in $(0,1)$, such that for every bounded measurable function $\phi : X \to \mathbb{R}$, and every Lipschitz continuous function $\psi : X \to \mathbb{R}$, we have for every integer $n \geq 1$

$$\left| \int_X \phi \circ f^n \cdot \psi \, d\nu - \int_X \phi \, d\nu \int_X \psi \, d\nu \right| \leq C \left(\sup_X |\phi| \right) \|\psi\|_{\mathrm{Lip}} \cdot \rho^n,$$

where $\|\psi\|_{\mathrm{Lip}} := \sup_{x,x' \in X, x \neq x'} \frac{|\psi(x) - \psi(x')|}{\mathrm{dist}(x,x')}$.

Our main result is stated for interval maps that are of class C^3 with non-flat critical points, see §2.2 for precisions. Moreover, we restrict the action of such a map f to its Julia set $J(f)$, which is also defined in §2.2. We denote by $\mathscr{A}$ the collection of interval maps of class C^3 with non-flat critical points, whose Julia set contains at least 2 points and is completely invariant.

For a map f in $\mathscr{A}$, our main result holds for a Hölder continuous potential $\varphi : J(f) \to \mathbb{R}$ that satisfies a certain bounded distortion hypothesis, stated precisely in §4. Roughly speaking, it requires that for each induced map F, defined through a sufficiently small nice couple as in [PRL07, PRL11], the induced potential is Hölder continuous. A potential $\varphi : J(f) \to \mathbb{R}$ satisfying this property is said to be "nicely distorted by f".

Main Theorem. *Let f be an interval map in $\mathscr{A}$ without neutral periodic points that is topologically exact on $J(f)$. Let $\varphi : J(f) \to \mathbb{R}$ be a Hölder continuous potential that is nicely distorted by f, and such that for some integer $n \geq 1$ the function $S_n(\varphi) := \sum_{j=0}^{n-1} \varphi \circ f^j$ satisfies*

$$\sup_{J(f)} \frac{1}{n} S_n(\varphi) < P(f|_{J(f)}, \varphi). \tag{1.3}$$

Then there is a unique equilibrium state ν of f for the potential φ. Moreover, the measure-theoretic entropy of ν is positive and ν is exponentially mixing for f. Finally, for every Hölder continuous function $\psi : J(f) \to \mathbb{R}$ that is nicely distorted by f, the function $t \mapsto P(f, \varphi + t\psi)$ is real analytic on a neighborhood of $t = 0$.

The following corollary is a direct consequence of the Main Theorem and the combination of [BRLSvS08, Theorem 1] and [RLS10, Theorem A], see Remark 4.2. Recall that for a differentiable map $f : I \to I$, a periodic point p of f of period n is *hyperbolic repelling*, if $|Df^n(p)| > 1$.

Corollary 1.1. *Let I be a compact interval, and let $f : I \to I$ be an interval map in $\mathscr{A}$ with all periodic points hyperbolic repelling that is topologically exact on I. Assume that for every critical value v of f we have*

$$\lim_{n \to +\infty} |Df^n(v)| = +\infty.$$

Then for every Hölder continuous potential $\varphi : I \to \mathbb{R}$ that is hyperbolic, there is a unique equilibrium state ν of f for the potential φ. Moreover, the measure-theoretic entropy of ν is positive, and ν is exponentially mixing for f. Finally, for every Hölder continuous function $\psi : I \to \mathbb{R}$, the function $t \mapsto P(\varphi + t\psi)$ is real analytic on a neighborhood of $t = 0$.

In [LRL12, Theorem A] we show that for a map f as in the corollary, the hypothesis that the potential φ is hyperbolic is in fact automatically satisfied. This implies that the conclusions of Corollary 1.1 hold for every pair of Hölder continuous functions φ and ψ.

The Main Theorem is closely related to the main result of [BT08], but there are some important differences. The first is that our Main Theorem applies to a larger class of maps: In [BT08, Theorem 4] the map f is assumed to be topologically exact on all of I; in our Main Theorem this corresponds to the special case where $J(f) = I$.

Perhaps the main improvement in our Main Theorem, in view of its applications in the companion paper [LRL12], is that the hypothesis that the potential φ satisfies the bounded range condition (1.1) in [BT08, Theorem 4] is replaced by the less restrictive hypothesis (1.3), that φ is hyperbolic. It is easy to see that every Hölder continuous potential φ satisfying the bounded range condition satisfies (1.3) with $n = 1$, and with $J(f)$ replaced by I. On the other hand, for every map f as in the Main Theorem there is a potential φ that satisfies (1.3) with $n = 1$, but that is not co-homologous to any continuous potential satisfying the bounded range condition, see Appendix A.

Let us also mention that [BT08, Theorem 4] has 2 hypotheses on the regularity of the potential. The first is that the potential has "small variations" for the partition generated by the partition of the interval domain into maximal intervals of monotonicity. Our hypothesis that the potential is Hölder continuous is stronger. The second hypothesis on the potential guarantees that for a suitable induced map the "induced potential" has bounded distortion; it is similar to our hypothesis that the potential is nicely distorted by the map.

1.2. **Organization.** To prove the Main Theorem we use the inducing scheme developed in [PRL07, PRL11]. In this construction, the induced map is defined through a "nice couple"; we recall the definition of these in §§2.3, 2.4, after some preliminary considerations in §§2.1, 2.2.

There are 2 main ingredients in the proof of the Main Theorem. The first is a technical estimate (Proposition A in §3), from which we easily deduce the exponential tail estimates (Corollary 3.1) required to apply Young's general result. In the proof of this estimate, which occupies all of §3, we use an estimate on pressure of the first landing domains to a nice set, which we state as Lemma 2.5 in §2.5.

The second main ingredient in the proof of the Main Theorem is a Bowen type formula: For every sufficiently small nice couple, a certain 2 variables pressure function of the induced map is finite on a neighborhood of $(0, P(f, \varphi))$, and it vanishes on the graph of $t \mapsto P(f, \varphi + t\psi)$. This is stated as Proposition B in §4. It is in the proof of this result that we use for the first time that φ and ψ are nicely distorted. The proof of Proposition B is similar to the considerations in [PRL11, §7], but it is simpler

thanks to our characterization of the pressure function in terms of diffeomorphic pull-backs (Lemma 4.4).

The proof of the Main Theorem is at the end of §4.3.

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2. Preliminaries

After fixing some notation and terminology in §2.1, in §2.2 introduce the class of interval maps considered in this paper, in §2.3 we recall the definition of nice sets and couples, and in §2.4 we recall the definition of the induced map associated to a nice couple. Finally, in §2.5 we make a technical estimate that is used in the proofs of Propositions A and B.

2.1. **Generalities.** We denote by $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ the extended real line.

For an interval J of $\mathbb{R}$, denote by $|J|$ its length. Moreover, for $\lambda > 0$ denote by λJ the open interval with the same midpoint as J and length $\lambda |J|$. Given $\tau > 0$, an interval J is *τ-well inside an interval $\widehat{J}$*, if $(1 + 2\tau)J$ is contained in $\widehat{J}$.

Throughout the rest of this article, fix a compact interval I of $\mathbb{R}$. We endow it with the distance dist induced by the norm distance of $\mathbb{R}$. For x in I and $r > 0$, denote by $B(x, r)$ the ball of I centered at x and radius r.

Given a differentiable map $f : I \to I$, denote by $\mathrm{Crit}(f)$ the subset of I on which the derivative of f vanishes. On the other hand, for each subset V of I and each integer $n \geq 1$, each connected component of $f^{-n}(V)$ is called a *pull-back of V by f^n*. A pull-back W of V by f^n is *diffeomorphic* if f^n maps W diffeomorphically onto a connected component of V, and it is *non-diffeomorphic* otherwise.

2.2. **Interval maps of class C^3 with non-flat critical points.** We introduce a class of interval maps that includes non-degenerate smooth maps as special cases.

A differentiable interval map $f : I \to I$ is *of class C^3 with non-flat critical points*, if it has a finite number of critical points and if:

- The map f is of class C^3 outside $\mathrm{Crit}(f)$;
- For each critical point c of f there exists a number $\ell_c > 1$ and diffeomorphisms ϕ and ψ of $\mathbb{R}$ of class C^3, such that $\phi(c) = \psi(f(c)) = 0$, and such that on a neighborhood of c on I, we have
$$|\psi \circ f| = |\phi|^{\ell_c}.$$

Note that each map of class C^3 with non-flat critical points is continuously differentiable.

Definition 2.1. Let $f : I \to I$ be an interval map of class C^3 with non-flat critical points. The *Julia set $J(f)$ of f* is the complement of the largest open subset of I on which the family of iterates of f is normal.

In contrast with the complex setting, the Julia set of an interval map of class C^3 with non-flat critical points might be empty, reduced to a single point, or might not be completely invariant.[‡] However, if the Julia set of such a map f is not completely invariant, then it is possible to make an arbitrarily small smooth perturbation of f outside a neighborhood of $J(f)$, so that the Julia set of the perturbed map is completely invariant and coincides with $J(f)$. We note also that if f has no neutral periodic point, then $J(f)$ is the complement of the basins of periodic attractors. For background on the theory of Julia sets, see for example [dMvS93].

As in the introduction, we denote by $\mathscr{A}$ the collection of interval maps of class C^3 with non-flat critical points, whose Julia set contains at least 2 points and is completely invariant. Moreover, for f in $\mathscr{A}$ we put

$$\mathrm{Crit}'(f) := \mathrm{Crit}(f) \cap J(f),$$

and restrict the action of f to $J(f)$. In particular, the topological pressure of f is defined through measures supported on $J(f)$, and equilibrium states are supported on $J(f)$.

2.3. **Nice sets and couples.** Let $f : I \to I$ be a map in $\mathscr{A}$. An open subset V of I is *a nice set for f*, if the following hold:

- Each connected component of V contains exactly one critical point of f in $J(f)$;
- $\overline{V}$ is disjoint from the forward orbits of critical points not in $J(f)$ and periodic orbits not in $J(f)$;
- For every integer $n \geq 0$ we have $f^n(\partial V) \cap V = \emptyset$.

In this case, for each c in $\mathrm{Crit}'(f)$ we denote by V^c the connected component of V containing c. For $\tau > 0$, a nice set V for f is *τ-nice*, if for each point x in V and each integer $n \geq 1$ such that $f^n(x)$ is in V, the pull-back of V by f^n containing x is τ-well inside a connected component of V. Moreover, a nice set V is *τ-well inside a nice set $\widehat{V}$*, if for each c in $\mathrm{Crit}'(f)$ the interval V^c is τ-well inside $\widehat{V}^c$. A *nice couple for f* is a pair of nice sets $(\widehat{V}, V)$ such that $\overline{V} \subset \widehat{V}$, and such that for every integer $n \geq 1$ the set $f^n(\partial V)$ is disjoint from $\widehat{V}$.

Lemma 2.2. *For every interval map $f : I \to I$ in $\mathscr{A}$ that is topologically exact on its Julia set, there is $\tau > 0$ such that the following property holds: For every $\delta > 0$ there is a nice couple $(\widehat{V}, V)$ such that $\widehat{V} \subset B(\mathrm{Crit}'(f), \delta)$ and such that V is τ-well inside $\widehat{V}$.*

This result is an easy consequence of the following.

Proposition 2.3 ([CL09], Proposition 5)**.** *For every interval map f in $\mathscr{A}$ that is topologically exact on its Julia set, there is $\tau > 0$ such that for every $\delta > 0$ there is a τ-nice set for f contained in $B(\mathrm{Crit}'(f), \delta)$.*

[‡]This last property can only happen if there is a turning point in the interior of the basin of a one-sided attracting neutral periodic point, that is eventually mapped to this neutral periodic point.

Proof of Lemma 2.2. Let $\mathscr{C}$ be the set of those c in $\mathrm{Crit}'(f)$ such that the closure of the set $\bigcup_{i=1}^{+\infty} f^i(c)$ intersects $\mathrm{Crit}'(f)$. In the case $\mathscr{C}$ is not all of $\mathrm{Crit}'(f)$, reducing δ if necessary assume that for each c in $\mathrm{Crit}'(f)$ that is not in $\mathscr{C}$, we have $\delta < \mathrm{dist}\left(\bigcup_{i=1}^{+\infty} f^i(c), \mathrm{Crit}'(f)\right)$.

Let τ be given by Proposition 2.3, and let $\widehat{V}$ be a τ-nice set for f contained in $B(\mathrm{Crit}'(f), \delta)$. Note that to construct a nice set V satisfying the desired properties, it is enough to find for each c in $\mathrm{Crit}'(f)$ an interval V^c containing c that is τ-well inside $\widehat{V}^c$, and such that for each integer $n \geq 1$ the set $f^n(\partial V^c)$ is disjoint from $\widehat{V}$. Suppose c in $\mathscr{C}$. Then there is an integer $n \geq 1$ such that $f^n(c)$ is in $\widehat{V}$; let n_0 be the minimal integer with this property, and let V^c be the pull-back of $\widehat{V}$ by f^{n_0} containing c. Then for every integer $n \geq 1$ the set $f^n(\partial V^c)$ is disjoint from $\widehat{V}$. Moreover, since $\widehat{V}$ is τ-nice, V^c is τ-well inside $\widehat{V}^c$. It remains to consider the case where c is not in $\mathscr{C}$. By our choice of δ, this implies that c is not contained in the closure of a connected component of the set

$$R(\widehat{V}) := \{z \in \widehat{V} : \text{ there is } n \geq 1 \text{ such that } f^n(z) \in \widehat{V}\}.$$

Therefore, we can find an interval V^c containing c that is τ-well inside $\widehat{V}^c$, and such that ∂V^c is disjoint from $R(\widehat{V})$. This last property implies that for every integer $n \geq 1$ the set $f^n(\partial V^c)$ is disjoint from $\widehat{V}$. This finishes the construction of the nice set $V := \bigcup_{c \in \mathrm{Crit}'(f)} V^c$ and completes the proof of the proposition. $\square$

2.4. **The canonical induced map associated to a nice couple.** Let f be a map in $\mathscr{A}$ and let $(\widehat{V}, V)$ be a nice couple for f. An integer $m \geq 1$ is a *good time for a point* x in V, if $f^m(x)$ is in V and if the pull-back of $\widehat{V}$ by f^m containing x is diffeomorphic. Denote by D the set of all those points in V having a good time. Moreover, for each x in D, denote by $m(x)$ the least good time of x. Note that $m(x)$ is constant on each component W of D; denote the common value by $m(W)$. The *canonical induced map associated to the nice couple* $(\widehat{V}, V)$ is the map $F : D \to V$ defined by $F(x) := f^{m(x)}(x)$. Note that for each connected component W of D the set $F(W)$ is a connected component of V; denote by $c(W)$ the critical point of f such that $F(W) = V^{c(W)}$. Denote by $J(F)$ the maximal invariant set of F, which is equal to the set of all those points in V having infinitely many good times. Moreover, for each integer $n \geq 1$ and each y in $F^{-n}(V)$, put

$$m_n(y) := m(y) + m(F(y)) + \cdots + m(F^{n-1}(y)).$$

Note that $m_n(\cdot)$ is constant on every connected component W of $F^{-n}(V)$; we denote the common value by $m_n(W)$.

When f is a complex rational map, the following lemma is [PRL07, Lemma 4.1]. The proof can be easily adapted to interval maps in $\mathscr{A}$.

Lemma 2.4. *Let f be a map in $\mathscr{A}$ that is topologically exact on $J(f)$. Then there is $\delta > 0$ such that for each nice couple $(\widehat{V}, V)$ satisfying*

$$\max_{c \in \mathrm{Crit}'(f)} |\widehat{V}^c| \leq \delta,$$

there is $\widetilde{c}$ in $\mathrm{Crit}'(f)$ such that the set

$$\{m(W) : W \text{ is a connected component of } D \cap V^{\widetilde{c}} \text{ such that } F(W) = V^{\widetilde{c}}\},$$

is non-empty and its greatest common divisor is equal to 1.

2.5. **First landing pressure.** Given an interval map $f : I \to I$ in $\mathscr{A}$, and a nice set V for f, let D_V^* be the set of those points x in $I \setminus V$ for which there is an integer $\ell \geq 1$ such that $f^\ell(x)$ is in V. For x in D_V^*, let $\ell(x)$ be the least integer $\ell \geq 1$ such that $f^\ell(x)$ is in V, and put $E_V(x) := f^{\ell(x)}(x)$. Note that for each x in D_V^*, the pull-back W of V by E_V containing x is diffeomorphic. Furthermore, W is a connected component of D_V^*, and the function ℓ is constant on W equal to $\ell(x)$; we put $\ell(W) := \ell(x)$. Denote by $\mathfrak{D}_V^*$ the collection of all the connected components of D_V^*.

This section is devoted to prove the following lemma, which is analogous to part 1 of [PRL11, Proposition 6.1].

Lemma 2.5. *Let f be an interval map in $\mathscr{A}$ without neutral periodic points, and let $\varphi : J(f) \to \mathbb{R}$ be a Hölder continuous potential. Then for every sufficiently small nice set V there is $p_0 < P(f, \varphi)$ such that*

$$\sum_{W \in \mathfrak{D}_V^*} \sup_{W \cap J(f)} \exp\left(S_{\ell(W)}(\varphi - p_0 \mathbb{1}_{J(f)})\right) < +\infty. \tag{2.1}$$

In particular, for each x_0 in V we have

$$\sum_{x \in E_V^{-1}(x_0)} \exp\left(S_{\ell(x)}(\varphi - p_0 \mathbb{1}_{J(f)})(x)\right) < +\infty.$$

To prove this lemma we use the following, which is essentially [PRL11, Lemma 6.2].

Lemma 2.6. *Let f be an interval map in $\mathscr{A}$. For every Hölder continuous function $\varphi : J(f) \to \mathbb{R}$, and every forward invariant set K such that f is uniformly expanding on K, we have*

$$P(f|_K, \varphi|_K) < P(f, \varphi).$$

Proof. Enlarging K if necessary, assume that the restriction of f to K admits a Markov partition, see for example [PU10, Theorem 4.5.2 and Remark 4.5.3]. So there is at least one equilibrium state ν of $f|_K$ for the potential $\varphi|_K$.

Enlarge K with more cylinders to obtain a compact forward invariant subset K' of $J(f)$, so that f restricted to K' admits a Markov partition, and so that the relative interior of K in K' is empty. It follows that ν

cannot be an equilibrium measure for $f|_{K'}$ for the potential $\varphi|_{K'}$, so we have

$$P(f|_K, \varphi|_K) = h_\nu(f) + \int \varphi d\nu < P(f|_{K'}, \varphi|_{K'}) \leq P(f, \varphi).$$

$\square$

Proof of Lemma 2.5. Let I be the domain of f. By the Koebe principle, for every sufficiently small nice couple $(\widehat{V}, V)$ for f, the following property holds: For every U in $\mathfrak{D}_V^*$, the distortion of $f^{\ell(U)}$ on U is bounded independently of U. Let V' be a sufficiently small neighborhood of $\mathrm{Crit}'(f)$ that is contained in V, and so that for each c in $\mathrm{Crit}'(f)$ there is a point x_c of V^c contained in

$$K(V') := \{x \in I : \text{ for every integer } n \geq 0 \text{ we have } f^n(x) \notin V'\}.$$

Since f has no neutral periodic point, a theorem of Mañé asserts that f is uniformly expanding on $K(V')$, see [Mañ85]. So by Lemma 2.6 we have $P(f, \varphi) > P(f|_{K(V')}, \varphi|_{K(V')})$. On the other hand, combined with the distortion property above, the fact that f is uniformly expanding on $K(V')$ implies that there is a constant $C > 0$ such that for every U in $\mathfrak{D}_V^*$, and every pair of points x and x' in U we have

$$|S_{\ell(U)}(\varphi)(x) - S_{\ell(U)}(\varphi)(x')| \leq C.$$

Since each connected component W of D_V^* is a diffeomorphic pull-back of V, there is a unique c in $\mathrm{Crit}'(f)$ and x_W in W, such that $f^{\ell(W)}(x_W) = x_c$; note that x_W is in $K(V')$. Then,

$$\begin{aligned}
\limsup_{\ell \to +\infty} \frac{1}{\ell} \log & \sum_{\substack{W \in \mathfrak{D}_V^* \\ \ell(W) = \ell}} \sup_{W \cap J(f)} \exp\left(S_\ell(\varphi)\right) \\
&= \limsup_{\ell \to +\infty} \frac{1}{\ell} \log \sum_{\substack{W \in \mathfrak{D}_V^* \\ \ell(W) = \ell}} \exp\left(S_\ell(\varphi)(x_W)\right) \\
&\leq \limsup_{\ell \to +\infty} \frac{1}{\ell} \log \sum_{c \in \mathrm{Crit}'(f)} \sum_{x \in K(V') \cap f^{-\ell}(x_c)} \exp\left(S_\ell(\varphi)(x)\right).
\end{aligned}$$

Together with a straight forward adaptation of [LRL12, Lemma 2.6] to $f|_{K(V')}$, the above implies

$$\limsup_{\ell \to +\infty} \frac{1}{\ell} \log \sum_{\substack{W \in \mathfrak{D}_V^* \\ \ell(W) = \ell}} \sup_{W \cap J(f)} \exp\left(S_\ell(\varphi)\right) \leq P(f|_{K(V')}, \varphi|_{K(V')}).$$

Thus, if we fix p_0 in $(P(f|_{K(V')}, \varphi|_{K(V')}), P(f, \varphi))$, then there is a constant $C' > 0$ such that for every $\ell \geq 1$ we have

$$\sum_{\substack{W \in \mathfrak{D}_V^* \\ \ell(W) = \ell}} \sup_{W \cap J(f)} \exp\left(S_\ell(\varphi) - p_0 \ell\right) \leq C' \exp\left(-\ell\left(p_0 - P(f|_{K(V')}, \varphi|_{K(V')})\right)\right).$$

Since the right hand side of this inequality is exponentially small in ℓ, this implies (2.1). $\square$

3. Exponential tail estimate

The purpose of this section is to prove Proposition A, below, from which we derive the exponential tail estimate (Corollary 3.1) that is used to apply the general result of Young.

The following is analogous to part 2 of [PRL07, Key Lemma]; in the statement we use the terminology introduced in §2.4.

Proposition A. *Let f be an interval map in $\mathscr{A}$ having no neutral periodic point, and let $\varphi : J(f) \to \mathbb{R}$ be a Hölder continuous potential satisfying $\sup_{J(f)} \varphi < P(f, \varphi)$. Then for every sufficiently small nice couple $(\widehat{V}, V)$ for f, there exist $p_1 < P(f, \varphi)$ such that the following property holds. Let $F : D \to V$ be the canonical induced map associated to $(\widehat{V}, V)$, and let $\mathfrak{D}$ be the collection of connected components of D. Then*

$$\sum_{W \in \mathfrak{D}} \sup_{W \cap J(f)} \exp(S_{m(W)}(\varphi - p_1 \mathbb{1}_{J(f)})) < +\infty.$$

The proof of this proposition is at the end of this section. First we derive from it Corollary 3.1, below.

Let $f : I \to I$ be an interval map in $\mathscr{A}$, let $(\widehat{V}, V)$ be a nice couple for f, and let $F : D \to V$ be the canonical induced map associated to $(\widehat{V}, V)$. Given a measurable function $G : J(F) \to \mathbb{R}$, a Borel measure μ on I is *G-conformal for F*, if it is supported on $J(F)$, and if for each connected component W of D, and each Borel subset U of $W \cap J(F)$, we have $\mu(F(U)) = \int_U G \, d\mu$.

Corollary 3.1. *Let f be an interval map in $\mathscr{A}$, and let $\varphi : J(f) \to \mathbb{R}$ be a Hölder continuous potential satisfying $\sup_{J(f)} \varphi < P(f, \varphi)$. Given a nice couple $(\widehat{V}, V)$ for f, consider the corresponding canonical induced map $F : D \to V$, define the function $\Phi^{\varphi} : J(F) \to \mathbb{R}$ by*

$$\Phi^{\varphi} := S_{m(\cdot)} \left(\varphi - P(f, \varphi) \mathbb{1}_{J(f)} \right),$$

and let μ be a $\exp(-\Phi^{\varphi})$-conformal measure for F. Then, provided the nice couple $(\widehat{V}, V)$ is sufficiently small, the following properties hold:

1. There is $\varepsilon_0 > 0$ such that

$$C' := \sum_{W \in \mathfrak{D}} \exp(m(W)\varepsilon_0) \, \mu(W) < +\infty.$$

In particular, for every integer $n \geq 1$ we have

$$\sum_{\substack{W \in \mathfrak{D} \\ m(W) \geq n}} \mu(W) \leq C' \exp(-n\varepsilon_0);$$

2. *For every continuous function $\psi : J(f) \to \mathbb{R}$, every pair of real numbers t and p, and every $\gamma > 0$, we have*
$$\int_{J(F)} \left| S_{m(\cdot)} \left(\varphi + t\psi - p\mathbb{1}_{J(f)} \right) \right|^{\gamma} \, d\mu < +\infty.$$

Proof. Let $(\widehat{V}, V)$ be a nice couple for f, for which the conclusion of Proposition A holds for some constant $p_1 < P(f, \varphi)$.

1. Put $\varepsilon_0 := P(f, \varphi) - p_1$ and note that by the conformality of μ, for every element W of $\mathfrak{D}$ we have
$$\mu(V^{c(W)}) = \mu(F(W \cap J(F))) = \int_{W \cap J(F)} \exp\left(-\Phi^{\varphi}\right) d\mu,$$
so
$$\mu(W) = \mu(W \cap J(F)) \le \mu(V^{c(W)}) \sup_{W \cap J(F)} \exp\left(\Phi^{\varphi}\right).$$
Therefore, by Proposition A we have
$$\sum_{W \in \mathfrak{D}} \exp\left(m(W)\varepsilon_0\right) \mu(W) \\ \le \mu(V) \sum_{W \in \mathfrak{D}} \sup_{W \cap J(F)} \exp\left(\Phi^{\varphi} + \varepsilon_0 m(W) \mathbb{1}_{J(f)}\right) < +\infty.$$
The proof of part 1 is thus complete.

2. Since for every pair of real numbers t and p, the function $\varphi + t\psi - p\mathbb{1}_{J(f)}$ is continuous, there is $M > 0$ such that $\sup_{J(f)} |\varphi + t\psi - p\mathbb{1}_{J(f)}| \le M$. Thus,
$$\int_{J(F)} \left| S_{m(\cdot)} \left(\varphi + t\psi - p\mathbb{1}_{J(f)} \right) \right|^{\gamma} d\mu \le \sum_{W \in \mathfrak{D}} (M m(W))^{\gamma} \mu(W) \\ \le M^{\gamma} \sum_{n=1}^{+\infty} \left(n^{\gamma} \sum_{W \in \mathfrak{D}, m(W) = n} \mu(W) \right).$$
Since by part 1 the sum $\sum_{W \in \mathfrak{D}, m(W)=n} \mu(W)$ is exponentially small in n, we conclude that $\int_{J(F)} \left| S_{m(\cdot)} \left(\varphi + t\psi - p\mathbb{1}_{J(f)} \right) \right|^{\gamma} d\mu$ is finite, as wanted. □

The rest of this section is devoted to the proof of Proposition A.

Definition 3.2. Let $f : I \to I$ be an interval map in $\mathscr{A}$, and let V be a nice set for f. Given an integer $m \ge 1$, a pull-back W of V by f^m is *bad of order* m, if for every integer m' in $\{1, \cdots, m\}$ such that $f^{m'}(W) \subset V$, the pull-back of V by $f^{m'}$ containing W is not diffeomorphic. Moreover, denote by $\mathfrak{B}_0(V)$ the collection of connected components of V, and for each integer $m \ge 1$, denote by $\mathfrak{B}_m(V)$ the collection of bad pull-backs of V by f^m.

Let f be an interval map in $\mathscr{A}$ and let $(\widehat{V}, V)$ be a nice couple for f. Denote by $\mathfrak{V}$ the collection of connected components of V, and extend the function ℓ defined in §2.5, as constant equal to 0 on V. Note that for each U

in $\mathfrak{D}_V^*$, the map $f^{\ell(U)}$ maps a neighborhood $\widehat{U}$ of U diffeomorphically onto the connected component of $\widehat{V}$ containing $f^{\ell(U)}(U)$. For each integer $\widetilde{m} \geq 0$ and each $\widetilde{Y}$ in $\mathfrak{B}_{\widetilde{m}}(\widehat{V})$, denote by $\mathfrak{D}_{\widetilde{Y}}$ the collection of all intervals W for which the following holds: There is an interval $\widehat{W}$ and U in $\mathfrak{D}_V^* \cup \mathfrak{V}$, such that $W \subset \widehat{W} \subset \widetilde{Y}$, $U \subset f(V)$, and such that $f^{\widetilde{m}+1}$ maps W diffeomorphically onto U, and $\widehat{W}$ onto $\widehat{U}$.

The following is [RLS10, Lemma 6.5], see also [PRL11, Lemma 3.4].

Lemma 3.3. *Let f be an interval map in $\mathscr{A}$ and let $(\widehat{V}, V)$ be a nice couple for f. Moreover, let $F : D \to V$ be the canonical induced map associated to $(\widehat{V}, V)$, and let $\mathfrak{D}$ be the collection of all the connected components of D. Then we have,*

$$\mathfrak{D} = \bigcup_{\widetilde{m}=0}^{+\infty} \bigcup_{\widetilde{Y} \in \mathfrak{B}_{\widetilde{m}}(\widehat{V})} \mathfrak{D}_{\widetilde{Y}},$$

and for each integer $\widetilde{m} \geq 0$, each $\widetilde{Y}$ in $\mathfrak{B}_{\widetilde{m}}(\widehat{V})$, and each x in $D \cap \widetilde{Y}$, we have

$$m(x) = \widetilde{m} + 1 + \ell(f^{\widetilde{m}+1}(x)). \tag{3.1}$$

Part 1 of the following lemma is similar to part 1 of [PRL07, Lemma 7.1].

Lemma 3.4. *Let f be an interval map in $\mathscr{A}$, and let V be a nice set for f. Let $L \geq 1$ be an integer such that for every critical point c and every i in $\{1, 2, \cdots, L\}$, we have either*

$$f^i(c) \notin V \text{ or } f^i(c) \in \mathrm{Crit}(f).$$

Then the following hold.

1. *For each integer $n \geq 1$, there are at most $((L+1)\#\,\mathrm{Crit}(f))^{1+\frac{n}{L}}$ bad pull-backs of V by f^n.*
2. *For each integer $n \geq 1$, each pull-back W of V by f^n, and each point y in V, we have*
$$\#(f^{-n}(y) \cap W) \leq 2^{1+n/L}.$$

Proof.

1. Fix $n \geq 1$. For a given non-diffeomorphic pull-back W of V by f^n define an integer $s \geq 1$ and a strictly increasing sequence of integers $(n_0, \cdots, n_s)$ with $n_0 = 0$ and $n_s = n$ by induction as follows. Suppose $j \geq 0$ is an integer such that $n_j \leq n-1$ is already defined. If $n_j + L \geq n$ or if $n_j + L \leq n-1$ and for each i in $\{n_j + L, \cdots, n-1\}$ the set $f^i(W)$ does not intersect $\mathrm{Crit}(f)$, then put $n_{j+1} := n, s := j+1$ and stop. Otherwise, let n_{j+1} be the least integer i in $\{n_j + L, \cdots, n-1\}$ such that $f^i(W) \cap \mathrm{Crit}(f) \neq \emptyset$.

Fix an integer $n \geq 1$. For every bad pull-back of V by f^n we associate a strictly increasing sequence $(n_0, \cdots, n_s)$ as above and associate a sequence of critical points $(c_1, \cdots, c_{s-1})$ such that $c_i \in f^{n_i}(W)$ for each $i \in \{1, \cdots, s-1\}$. Note that, by definition of L and $(n_0, \cdots, n_s)$, each pair of sequences

$(n_0, \cdots, n_s)$ and $(c_1, \cdots, c_{s-1})$ can have at most $\#\operatorname{Crit}(f)$ bad pull-backs of order n. It follows that there are at most $\#\operatorname{Crit}(f)^s$ bad pull-backs of V by f^n with the same sequence $(n_0, \cdots, n_s)$.

On the other hand, by our construction of $(n_0, \cdots, n_s)$ we have $n_i - n_{i-1} \geq L$ for every integer $i \in \{1, \cdots, s-1\}$. Hence $s \leq \frac{n}{L} + 1$ and for each integer $m \in \{1, \cdots, n\}$ there is at most one integer $r \in \{0, 1, \cdots, L-1\}$ such that $m + r$ is one of the n_i. It follows that there are at most $(L+1)^{1+n/L}$ such increasing sequences $(n_0, \cdots, n_s)$. Therefore we conclude that the total number of bad pull-backs of V by f^n is at most

$$\#\operatorname{Crit}(f)^s (L+1)^{1+n/L} \leq (\#\operatorname{Crit}(f)(L+1))^{1+n/L}.$$

2. Fix $y \in V$ and a pull-back W of V by f^n. If W is a diffeomorphic pull-back of V by f^n, then $\#(f^{-n}(y) \cap W) = 1 \leq 2^{1+n/L}$. Now we assume W is a non-diffeomorphic pull-back of V by f^n. As in part 1, we associate a strictly increasing sequence $(n_0, \cdots, n_s)$ for W. Using the definitions of L and $(n_0, \cdots, n_s)$ again, we know that for every $i \in \{1, \cdots, s\}$ the map $f^{n_i - n_{i-1}}$ has at most one critical point in $f^{n_i}(W)$. It follows that the map f^n has at most s critical points in W. Noticing that $s \leq 1 + n/L$ as part 1, we conclude that

$$\#(f^{-n}(y) \cap W) \leq 2^s \leq 2^{1+n/L}.$$

□

Proof of Proposition A. Fix $\eta_0 > 0$ satisfying $\eta_0 < P(f, \varphi) - \sup_{J(f)} \varphi$, and let $L \geq 1$ be large enough so that

$$(2(L+1)\#\operatorname{Crit}(f))^{1/L} < \exp(\eta_0). \tag{3.2}$$

Let $(\widehat{V}, V)$ be a sufficiently small nice couple for f, such that for every i in $\{1, 2, \cdots, L\}$ and c in $\operatorname{Crit}'(f)$ we have either

$$f^i(c) \notin \widehat{V} \text{ or } f^i(c) \in \operatorname{Crit}(f).$$

Taking $(\widehat{V}, V)$ smaller if necessary, assume that the conclusion of Lemma 2.5 is satisfied for some p_1 in $\left(\sup_{J(f)} \varphi + \eta_0, P(f, \varphi)\right)$.

Note that for every x in I there are at most $\#\operatorname{Crit}(f) + 1$ points in $f^{-1}(x)$, and that for every W in $\mathfrak{D}_0 := \bigcup_{c \in \operatorname{Crit}'(f)} \mathfrak{D}_{V^c}$, the set $f(W)$ is in $\mathfrak{D}_V^*$. So, using that φ is continuous and hence bounded, we have by Lemma 2.5

$$A := \sum_{W \in \mathfrak{D}_0} \sup_{W \cap J(f)} \exp(S_{m(W)}(\varphi - p_1 \mathbb{1}_{J(f)})) < +\infty.$$

In view of Lemma 3.3, to complete the proof of the lemma it is enough to show

$$B := \sum_{n=1}^{+\infty} \sum_{\widetilde{W} \in \mathfrak{B}_n(\widehat{V})} \sum_{W \in \mathfrak{D}_{\widetilde{W}}} \sup_{W \cap J(f)} \exp(S_{m(W)}(\varphi - p_1 \mathbb{1}_{J(f)})) < +\infty. \tag{3.3}$$

Let $n \geq 1$ be an integer, and let $\widetilde{W}$ be in $\mathfrak{B}_n(\widehat{V})$. By part 2 of Lemma 3.4, for every x in V there are at most $2^{(1+n/L)}$ points of $f^{-n}(x)$ in $\widetilde{W}$. Hence, noticing that $f^n(W)$ is in $\mathfrak{D}_0$, we have by (3.1)

$$\sum_{W \in \mathfrak{D}_{\widetilde{W}}} \sup_{W \cap J(f)} \exp(S_{m(W)}(\varphi - p_1 \mathbb{1}_{J(f)})) \leq 2^{1+\frac{n}{L}} \exp\left(n\left(\sup_{J(f)} \varphi - p_1\right)\right) \cdot A.$$

On the other hand, by part 1 of Lemma 3.4 the number of bad pull-backs of $\widehat{V}$ by f^n is bounded by $((L+1)\# \operatorname{Crit})^{1+n/L}$. Therefore, letting

$$M_n := ((L+1)\# \operatorname{Crit})^{1+n/L}\, 2^{1+\frac{n}{L}} \exp\left(n\left(\sup_{J(f)} \varphi - p_1\right)\right),$$

we have

$$\sum_{\widetilde{W} \in \mathfrak{B}_n(\widehat{V})} \sum_{W \in \mathfrak{D}_{\widetilde{W}}} \sup_{W \cap J(f)} \exp(S_{m(W)}(\varphi - p_1 \mathbb{1}_{J(f)})) \leq M_n \cdot A.$$

Noticing that $\kappa := \exp\left(\eta_0 + \sup_{J(f)} \varphi - p_1\right)$ is in $(0,1)$, we have by (3.2)

$$B \leq A \sum_{n=1}^{+\infty} M_n \leq 2A(L+1)(\# \operatorname{Crit}(f)) \sum_{n=1}^{+\infty} \kappa^n < +\infty.$$

This completes the proof of the proposition. $\square$

4. Proof of the Main Theorem

Throughout this section, fix a map $f : I \to I$ in $\mathscr{A}$. Following the general strategy in [PRL11], in §4.1 we consider a 2 variables "induced pressure function", and in §4.2 we show this function vanishes precisely on the graph of the pressure function of f (Proposition B). This is used in §4.3 to prove the Main Theorem.

4.1. **Two variables pressure function.** Let $(\widehat{V}, V)$ be a nice couple for f, and let $F : D \to V$ be the canonical induced map associated to $(\widehat{V}, V)$. Denote by $\mathfrak{D}$ the collection of connected components of D, and for each c in $\operatorname{Crit}'(f)$ denote by $\mathfrak{D}^c$ the collection of all elements of $\mathfrak{D}$ contained in V^c, so that $\mathfrak{D} = \bigsqcup_{c \in \operatorname{Crit}'(f)} \mathfrak{D}^c$. A word on the alphabet $\mathfrak{D}$ is *admissible*, if for every pair of consecutive letters W and W' we have $W \in \mathfrak{D}^{c(W')}$. For each integer $n \geq 1$, denote by E^n the collection of all admissible words of length n. Given W in $\mathfrak{D}$, denote by ϕ_W the inverse of $F|_W$. For an integer $n \geq 1$ and $\underline{W}$ in E^n, put

$$c(\underline{W}) := c(W_n) \text{ and } m(\underline{W}) := m(W_1) + \cdots + m(W_n).$$

Note that the composition

$$\phi_{\underline{W}} := \phi_{W_1} \circ \cdots \circ \phi_{W_n}$$

is defined on $V^{c(\underline{W})}$, and takes images in D; put $D_{\underline{W}} := \phi_{\underline{W}}(V^{c(W)})$. Moreover, $\phi_{\underline{W}}$ admits an extension to $\widehat{V}^{c(\underline{W})}$, that is a diffeomorphism onto its image, and maps $\widehat{V}^{c(\underline{W})}$ into V.

A direct application of the Koebe principle, as stated for example in [BRLSvS08], shows that F determines a Conformal Graph Directed Markov System (CGDMS) in the sense of [MU03], see for example [PRL07, §A.1].

Throughout the rest of this section we fix 2 continuous functions φ, $\psi : J(f) \to \mathbb{R}$, and for each pair of real numbers t and p, put

$$\phi_{t,p} := \varphi + t\psi - p\mathbb{1}_{J(f)}.$$

Consider the function $m(\cdot)$ defined in §2.4, that to each point x in D assigns the least good time $m(x)$ of x, and denote by $\Phi_{t,p}$ the function $S_{m(\cdot)}(\phi_{t,p}) : J(F) \to \mathbb{R}$. Note that $\Phi_{0,p}$ does not depend on ψ. Given t in $\mathbb{R}$ an integer $n \geq 1$, for each $\underline{W}$ in E^n put

$$V_t(\underline{W}) := \sup_{D_{\underline{W}} \cap J(F)} (\Phi_{t,0}) - \inf_{D_{\underline{W}} \cap J(F)} (\Phi_{t,0}),$$

and

$$V_t(n) := \sup_{\underline{W} \in E^n} V_t(\underline{W}).$$

Using the terminology of [MU03], for t in $\mathbb{R}$ the function $\Phi_{t,0}$ *defines a Hölder continuous potential on the symbolic space associated to* F, if $V_t(n)$ is exponentially small in n. In this case, for each p in $\mathbb{R}$ the function $\Phi_{t,p}$ defines a Hölder continuous potential on the symbolic space associated to F.

Definition 4.1. For a map f in $\mathscr{A}$, a function $\varphi : J(f) \to \mathbb{R}$ is *nicely distorted by* f, if for every sufficiently small nice couple $(\widehat{V}, V)$ for f the function $S_{m(\cdot)}(\varphi)|_{J(F)}$, denoted by $\Phi_{0,0}$ above, defines a Hölder continuous function on the symbolic space associated to F.

Note that if φ and ψ are both nicely distorted by f, then there are $C > 0$ and θ in $(0,1)$ such that for every t in $\mathbb{R}$ and every integer $n \geq 1$, we have $V_t(n) \leq C(1 + |t|)\theta^n$. In particular, for every t and p in $\mathbb{R}$ the function $\Phi_{t,p}$ defines a Hölder continuous potential on the symbolic space associated to F.

Remark 4.2. For $\beta > 0$, a map f in $\mathscr{A}$ satisfies the *Polynomial Shrinking Condition with exponent* β, if there exist constants $\rho_0 > 0$ and $C_0 > 0$ such that for every x in $J(f)$, every integer $n \geq 1$, and every connected component W of $f^{-n}(B(x, \rho))$, we have $\operatorname{diam}(W) \leq Cn^{-\beta}$. Since the canonical induced map associated to a nice couple is uniformly expanding, it is clear that if for some $\beta > 1$ the map f satisfies the Polynomial Shrinking Condition with exponent β, then every Hölder continuous function $\varphi : J(f) \to \mathbb{R}$ of exponent in $(\beta^{-1}, 1]$ is nicely distorted by f, see for example [BT08, Lemma 3 b)]. Thus [BRLSvS08, Theorem 1] and [RLS10, Theorem A] imply that if f is an interval map in $\mathscr{A}$ satisfying the hypotheses of Corollary 1.1, then every Hölder continuous potential is nicely distorted for f.

Given t and p in $\mathbb{R}$, for each integer $n \geq 1$ put

$$Z_n(\Phi_{t,p}) := \sum_{\underline{W} \in E^n} \exp \left(\sup_{D_{\underline{W}} \cap J(F)} \sum_{j=0}^{n-1} \Phi_{t,p} \circ F^j \right).$$

The sequence $\{\log Z_n(\Phi_{t,p})\}_{n=1}^{+\infty}$ is clearly sub-additive, so

$$P(F, \Phi_{t,p}) := \inf \left\{ \frac{1}{n} \log Z_n(\Phi_{t,p}) : n \geq 1 \right\} = \lim_{n \to +\infty} \frac{1}{n} \log Z_n(\Phi_{t,p}). \tag{4.1}$$

The number (4.1) is the *pressure function of F for the potential $\Phi_{t,p}$*. The function $\mathscr{P} : \mathbb{R}^2 \to \overline{\mathbb{R}}$ defined by

$$\mathscr{P}(t,p) := P(F, \Phi_{t,p}),$$

is important in what follows. It is easy to see that for each real number t, the function $p \mapsto \mathscr{P}(t,p)$ is non-increasing, and that it is continuous and strictly decreasing on the set where it is finite.

Lemma 4.3. *Let f be an interval map in $\mathscr{A}$, and let φ, $\psi : J(f) \to \mathbb{R}$ be continuous functions that are nicely distorted by f. Then for every sufficiently small nice couple $(\widehat{V}, V)$ for f, the function $\mathscr{P}$ defined above satisfies the following properties.*

1. *$\mathscr{P}$ is strictly negative on*

$$\left\{ (t,p) \in \mathbb{R}^2 : p > P(f, \varphi + t\psi) \right\}. \tag{4.2}$$

2. *$\mathscr{P}$ is real analytic on the interior of the set where it is finite.*

Proof. Assume $(\widehat{V}, V)$ is sufficiently small so that there are $C > 0$ and θ in $(0,1)$, such that for every t in $\mathbb{R}$ and every integer $n \geq 1$, we have $V_t(n) \leq C(1 + |t|)\theta^n$. In particular, for every pair of real numbers t and p the function $\Phi_{t,p}$ defines a Hölder continuous function on the symbolic space associated to F, and for every integer $k \geq 1$, every $\underline{W}$ in E^k, and every x and x' in $D_{\underline{W}} \cap J(F)$, we have

$$\left| \sum_{j=0}^{k-1} \Phi_{t,p} \circ F^j(x) - \sum_{j=0}^{k-1} \Phi_{t,p} \circ F^j(x') \right| \leq C(1 + |t|)(1 - \theta)^{-1}.$$

1. Let (t,p) in $\mathbb{R}^2$ be such that $p > P(f, \phi_{t,0})$, and put $C_0 := \exp(C(1 + |t|)(1-\theta)^{-1})$. Moreover, for each c in $\mathrm{Crit}'(f)$, choose a point z_c in $V^c \cap J(F)$.

Then we have

$$\begin{aligned}\sum_{k=1}^{+\infty} Z_k(\Phi_{t,p}) &= \sum_{k=1}^{+\infty} \sum_{\underline{W}\in E^k} \exp\left(\sup_{D_{\underline{W}}\cap J(F)} \sum_{j=0}^{k-1} \Phi_{t,p}\circ F^j\right)\\ &\le C_0 \sum_{k=1}^{+\infty} \sum_{c\in \mathrm{Crit}'(f)} \sum_{y\in F^{-k}(z_c)} \exp\left(S_{m_k(y)}(\phi_{t,p})(y)\right)\\ &\le C_0 \sum_{c\in\mathrm{Crit}'(f)} \sum_{n=1}^{+\infty} \exp(-pn) \sum_{y\in f^{-n}(z_c)} \exp\left(S_n(\phi_{t,0})(y)\right).\end{aligned}$$

Combined with [LRL12, Lemma 2.6], this implies

$$\sum_{k=1}^{+\infty} Z_k(\Phi_{t,p}) < +\infty,$$

and therefore that $\mathscr{P}(t,p) \le 0$. This shows that the function $\mathscr{P}$ is non-positive on (4.2). Since for each real number t the function $p \mapsto \mathscr{P}(t,p)$ is strictly decreasing on the set where it is finite, it follows that $\mathscr{P}$ is strictly negative on (4.2).

2. For each (t,p) in $\mathbb{R}^2$ such that $\mathscr{P}(t,p)$ is finite, [MU03, Proposition 2.1.9] implies that the function $\Phi_{t,p}$ defines a summable function on the symbolic space associated to F, in the sense of [MU03]. Since for each integer $n \ge 1$ we have $V_t(n) \le C(1+|t|)\theta^n$, the desired result is given by [MU03, Theorem 2.6.12]. □

4.2. **A Bowen type formula.** This section is devoted to the proof of the following proposition.

Proposition B. *Let f be an interval map in $\mathscr{A}$ without neutral periodic points, and such that f is topologically exact on $J(f)$. Moreover, let $\varphi, \psi : J(f) \to \mathbb{R}$ be Hölder continuous and nicely distorted by f, and suppose* $\sup_{J(f)} \varphi < P(f,\varphi)$. *Then for every sufficiently small nice couple $(\widehat{V}, V)$, the pressure function $\mathscr{P}$ defined in §4.1 is finite and real analytic on a neighborhood of $(t,p) = (0, P(f,\varphi))$, and there exists $\varepsilon_0 > 0$ such that for each t in $(-\varepsilon_0, \varepsilon_0)$ the function $\mathscr{P}$ vanishes at $(t, P(f, \varphi + t\psi))$.*

The proof of this proposition is given after the following lemma.

Given an integer $n \ge 1$ and a point x in I, a preimage y of x by f^n is *critical* if $Df^n(y) = 0$, and it is *non-critical* otherwise.

Lemma 4.4. *Let f be an interval map in $\mathscr{A}$ that is topologically exact on $J(f)$, and let $\phi : J(f) \to \mathbb{R}$ be a Hölder continuous potential satisfying* $\sup_{J(f)} \phi < P(f,\phi)$. *Then for every point x_0 of $J(f)$ having infinitely many non-critical preimages, there is $\delta > 0$ such that the following property holds: If for each integer $n \ge 1$ we denote by $\mathfrak{D}_n$ the collection of*

diffeomorphic pull-backs of $B(x_0, \delta)$ *by* f^n*, then*

$$\limsup_{n \to +\infty} \frac{1}{n} \log \sum_{W \in \mathfrak{D}_n} \inf_{W \cap J(f)} \exp(S_n(\phi)) \geq P(f, \phi).$$

The proof of this lemma is based on Przytycki and Urbański's adaptation to one-dimensional maps of Katok-Pesin theory, see [PU10, §11.6]. The proofs in [PU10, §11.6] are written for complex rational maps, but they apply without change to interval maps in $\mathscr{A}$.

Proof. In view of [Mañ85], the hypothesis that f is topologically exact on $J(f)$ implies that the repelling periodic points of f are dense in $J(f)$. It follows that there is a repelling periodic point p_0 of f such that all of its preimages are non-critical. Let $\delta_0 > 0$ be sufficiently small so that for every $\delta_* > 0$ there is $n_* \geq 1$ such that for every integer $n \geq n_*$ satisfying $f^n(p_0) = p_0$, the pull-back of $B(p_0, \delta_0)$ by f^n containing p_0 is diffeomorphic and contained in $B(p_0, \delta_*)$.

In part 1 below we prove the lemma with $\delta = \delta_0$, in the special case $x_0 = p_0$. In part 2 we deal with the general case using this special case.

1. Let $\varepsilon > 0$ be given. Since the measure-theoretic entropy of f is upper semi-continuous as a function defined on $\mathcal{M}(J(f), f)$, there is an equilibrium state ν of f for the potential ϕ. Replacing ν by one of its ergodic components is necessary, assume ν ergodic. We thus have

$$h_\nu(f) = P(f, \phi) - \int \phi \, d\nu \geq P(f, \phi) - \sup_{J(f)} \phi > 0,$$

and then Ruelle's inequality implies that the Lyapunov exponent of ν is strictly positive, see [Rue78]. By [PU10, Theorem 11.6.1], it follows that there is a compact and forward invariant subset X of $J(f)$ on which f is topologically transitive, so that $f : X \to X$ is open and uniformly expanding, and so that

$$P(f|_X, \phi|_X) \geq P(f, \phi) - \varepsilon.$$

Hence, if we fix x_0' in X, then there is $\delta' > 0$ such that the following property holds: For each integer $n \geq 1$ denote by $\mathfrak{D}_n'$ the collection of diffeomorphic pull-backs of $B(x_0', \delta')$ by f^n along X; then

$$\limsup_{n \to +\infty} \frac{1}{n} \log \sum_{W' \in \mathfrak{D}_n'} \inf_{W' \cap J(f)} \exp(S_n(\phi)) \geq P(f, \phi) - \varepsilon, \tag{4.3}$$

see for example [PU10, Proposition 4.4.3]. Since f is topologically exact on $J(f)$, there is an integer $n_0 \geq 1$ and a preimage p_0' of p_0 by f^{n_0} in $B(x_0', \delta')$. By our choice of p_0, the derivative $Df^{n_0}(p_0')$ is non-zero. So there is $\delta'' > 0$ such that the pull-back of $B(p_0, \delta'')$ by f^{n_0} containing p_0' is diffeomorphic and contained in $B(x_0', \delta')$. Let $m_0 \geq 1$ be an integer such that $f^{m_0}(p_0) = p_0$ and such that the pull-back of $B(p_0, \delta_0)$ containing p_0 is diffeomorphic and contained in $B(p_0, \delta'')$. It follows that the pull-back of $B(p_0, \delta_0)$ by $f^{m_0+n_0}$ containing p_0' is diffeomorphic and contained

in $B(x_0', \delta')$. Therefore, if for each integer $n \ge 1$ we denote by $\mathfrak{D}_n^0$ the collection of all diffeomorphic pull-backs of $B(p_0, \delta_0)$ by f^n, then by (4.3) we have

$$\limsup_{n \to +\infty} \frac{1}{n} \log \sum_{W \in \mathfrak{D}_n^0} \inf_{W \cap J(f)} \exp(S_n(\phi)) \ge P(f, \phi) - \varepsilon.$$

Since ε is an arbitrary positive number, this proves the desired inequality with $\delta = \delta_0$, in the special case $x_0 = p_0$.

2. The hypothesis that x_0 has infinitely many non-critical preimages implies that there is a non-critical preimage x_0' of x_0 such that all preimages of x_0' are non-critical. Since f is topologically exact on $J(f)$, there is a preimage of x_0' in $B(p_0, \delta_0)$, and therefore there is an integer $n \ge 1$ and a non-critical preimage x_0'' of x_0 by f^n that is in $B(p_0, \delta_0)$. It follows that there is $\delta > 0$ such that the pull-back of $B(x_0, \delta)$ by f^n that contains x_0'' is contained in $B(p_0, \delta_0)$. Then the desired assertion follows from the special case shown in part 1. □

Proof of Proposition B. We use the notations of §4.1.

Let $\delta > 0$ be sufficiently small so that the conclusion of Lemma 4.4 holds for every c_0 in $\mathrm{Crit}'(f)$ having infinitely many non-critical preimages. Let $(\widehat{V}, V)$ be a nice couple for f such that $\widehat{V} \subset \widetilde{B}(\mathrm{Crit}'(f), \delta)$, and such that the conclusion of Lemma 4.3 hold. Replacing $(\widehat{V}, V)$ by a smaller nice couple if necessary, assume by Lemma 2.5 and Proposition A that there is $p_1 < P(f, \varphi)$ such that

$$\sum_{W \in \mathfrak{D}_V^*} \sup_{W \cap J(f)} \exp(S_{\ell(W)}(\varphi - p_1 \mathbb{1}_{J(f)})) < +\infty, \tag{4.4}$$

and

$$\sum_{W \in \mathfrak{D}} \sup_{W \cap J(f)} \exp(S_{m(W)}(\varphi - p_1 \mathbb{1}_{J(f)})) < +\infty. \tag{4.5}$$

Fix p_2 in $(p_1, P(f, \varphi))$, and let $\varepsilon_0 > 0$ be small enough so that

$$\varepsilon_0 \sup_{J(f)} |\psi| < \min\{p_2 - p_1, P(f, \varphi) - p_2\}.$$

Note that by our choice of ε_0, for every t in $(-\varepsilon_0, \varepsilon_0)$ we have on $J(f)$ that

$$\varphi + t\psi > \varphi - (P(f, \varphi) - p_2) \mathbb{1}_{J(f)},$$

so

$$\begin{aligned} &P(f, \varphi + t\psi) \\ &\quad \ge P(f, \varphi - (P(f, \varphi) - p_2)\mathbb{1}_{J(f)}) = P(f, \varphi) - (P(f, \varphi) - p_2) = p_2. \end{aligned} \tag{4.6}$$

In part 1 below we prove that $\mathscr{P}$ is finite on the neighborhood $S := (-\varepsilon_0, \varepsilon_0) \times (p_2, +\infty)$ of $(0, P(f, \varphi))$. Then part 2 of Lemma 4.3 implies that $\mathscr{P}$ is real analytic on a neighborhood of $(t, p) = (0, P(f, \varphi))$. In part 2 we complete the proof of the proposition by showing that for every t in $(-\varepsilon_0, \varepsilon_0)$, the function $\mathscr{P}$ vanishes at $(t, P(f, \varphi + t\psi))$.

1. By our choice of ε_0, for every (t,p) in S we have on $J(f)$ that

$$\phi_{t,p} = \varphi + t\psi - p\mathbb{1}_{J(f)} \le \varphi + (p_2 - p_1 - p)\mathbb{1}_{J(f)} < \varphi - p_1\mathbb{1}_{J(f)}. \tag{4.7}$$

It follows by (4.5) that

$$Z_1(\phi_{t,p}) = \sum_{W\in\mathfrak{D}} \sup_{W\cap J(F)} \exp(S_{m(W)}(\phi_{t,p})) \le \sum_{W\in\mathfrak{D}} \sup_{W\cap J(F)} \exp(S_{m(W)}(\varphi - p_1\mathbb{1}_{J(f)})) < +\infty.$$

Noting that for every integer $n \ge 1$ we have $Z_n(\Phi_{t,p}) \le Z_1(\Phi_{t,p})^n$, this implies that $\mathscr{P}(t,p)$ is finite, and completes the proof that $\mathscr{P}$ is finite on S.

2. Fix t in $(-\varepsilon_0, \varepsilon_0)$ and for each c in $\mathrm{Crit}'(f)$ choose a point z_c in $V^c \cap J(F)$. By part 1, the function $p \mapsto \mathscr{P}(t,p)$ is finite, continuous, and strictly decreasing on $(p_2, +\infty)$. In view of part 1 of Lemma 4.3, it is enough to prove that for each p in $(p_2, P(f, \phi_{t,0}))$ we have $\mathscr{P}(t,p) \ge 0$. Suppose by contradiction that there is such a p satisfying $\mathscr{P}(t,p) < 0$. For each integer $n \ge 1$ denote by $\mathscr{G}_n$ the set of all those points x in $f^{-n}(\mathrm{Crit}'(f))$ such that the pull-back of V by f^n containing x is diffeomorphic. In view of Lemma 4.4, to obtain a contradiction it is enough to show that the sum

$$\sum_{n=1}^{+\infty} \sum_{x\in\mathscr{G}_n} \exp\left(S_n(\phi_{t,p})(x)\right) \tag{4.8}$$

is finite. By definition of $\mathscr{P}$, the hypothesis $\mathscr{P}(t,p) < 0$ implies that the sum

$$\sum_{k=1}^{+\infty} Z_k(\Phi_{t,p}) = \sum_{k=1}^{+\infty} \sum_{\underline{W}\in E^k} \sup_{D_{\underline{W}}\cap J(F)} \exp\left(S_{m(\underline{W})}(\phi_{t,p})\right)$$

is finite. Since $\Phi_{t,p}$ defines a Hölder continuous function on the symbolic space associated to F, it follows that for every c in $\mathrm{Crit}'(f)$ the sum

$$I(c) := \sum_{k=1}^{+\infty} \sum_{y\in F^{-k}(z_c)} \exp\left(\sum_{j=0}^{k-1} \Phi_{t,p} \circ F^j(y)\right)$$

is finite. On the other hand, if for each z_0 in V we put

$$L(z_0) := 1 + \sum_{z\in E_V^{-1}(z_0)} \exp\left(S_{\ell(z)}(\phi_{t,p})(z)\right),$$

then by (4.4) and (4.7), the supremum $\sup_V L$ is finite. Note that for each integer $n \ge 1$ and each point x in $\mathscr{G}_n$, either $E_V(x)$ is in $\mathrm{Crit}'(f)$ or there is an integer $k \ge 1$ such that $E_V(x)$ is in the domain of F^k and $n =$

$\ell(x) + m_k(E_V(x))$. Therefore, (4.8) is bounded from above by

$$\sum_{c \in \mathrm{Crit}'(f)} L(z_c) + \sum_{c \in \mathrm{Crit}'(f)} \sum_{k=1}^{+\infty} \sum_{y \in F^{-k}(z_c)} \exp\left(\sum_{j=0}^{k-1} \Phi_{t,p} \circ F^j(y)\right) L(y)$$
$$\leq \left(\sup_V L\right)\left(\# \mathrm{Crit}'(f) + \sum_{c \in \mathrm{Crit}'(f)} I(c)\right),$$

which is finite by the considerations above. We thus obtain a contradiction that completes the proof that $\mathscr{P}$ vanishes at $(t, P(f, \phi_{t,0}))$. $\square$

4.3. **Proof of the Main Theorem.** By hypothesis, there is an integer $n \geq 1$ such that the function $\widetilde{\varphi} := \frac{1}{n} S_n(\varphi)$ satisfies $\sup_{J(f)} \widetilde{\varphi} < P(f, \varphi)$. Since f is Lipschitz as a self-map of I, it follows that $\widetilde{\varphi}$ and $\widetilde{\psi} := \frac{1}{n} S_n(\psi)$ are both Hölder continuous. On the other hand, note that if $h : J(f) \to \mathbb{R}$ is the continuous function defined by $h := -\frac{1}{n} \sum_{j=0}^{n-1} (n-1-j) \varphi \circ f^j$, then $\varphi = \widetilde{\varphi} + h - h \circ f$. This implies that for every invariant measure ν of f we have $\int \widetilde{\varphi}\, d\nu = \int \varphi\, d\nu$, so $P(f, \widetilde{\varphi}) = P(f, \varphi)$, and φ and $\widetilde{\varphi}$ share the same equilibrium states. A similar argument shows that for every t in $\mathbb{R}$ we have $P(f, \widetilde{\varphi} + t\widetilde{\psi}) = P(f, \varphi + t\psi)$. Moreover, our hypothesis that φ and ψ are nicely distorted by f implies that $\widetilde{\varphi}$ and $\widetilde{\psi}$ are also nicely distorted by f. Thus, replacing φ by $\widetilde{\varphi}$ and ψ by $\widetilde{\psi}$ if necessary, we can assume $\sup_{J(f)} \varphi < P(f, \varphi)$.

For the rest of the proof, we proceed in a similar way as in the proof of [PRL11, Theorem A]. Denote by I the domain of f. In view of Lemma 2.2, we can find a nice couple $(\widehat{V}, V)$ for f for which the conclusions of Lemma 2.4 and Propositions A and B hold for some $\widetilde{c}$ in $\mathrm{Crit}'(f)$, some $p_1 < P(f, \varphi)$, and some $\varepsilon_0 > 0$, respectively. Replacing $(\widehat{V}, V)$ by a smaller nice couple if necessary, assume that the conclusions of Corollary 3.1 hold for $(\widehat{V}, V)$, and that the function $\Phi_{0,0}$ defines a Hölder continuous function on the symbolic space associated to the canonical induced map $F : D \to V$ associated to $(\widehat{V}, V)$. This implies that the function $\Phi^{\varphi} := \Phi_{0, P(f,\varphi)}$ has the same property.

By Proposition B, the function $\mathscr{P}$ vanishes at $(t, p) = (0, P(f, \varphi))$, so $P(F, \Phi^{\varphi}) = 0$. On the other hand, Proposition A implies that Φ^{φ} is "summable" for F in the sense of [MU03]. Thus [MU03, Proposition 4.2.5 and Theorem 3.2.3] imply that F admits a $\exp(-\Phi^{\varphi})$-conformal measure μ, as defined in §3.

To construct the equilibrium state for f, remark that, by standard considerations, F has an invariant measure ρ that is absolutely continuous with respect to the $\exp(-\Phi^{\varphi})$-conformal measure μ of F, see for example [Gou04,

§1], or [MU03, §6] in the case V is connected. The measure

$$\widehat{\rho} := \sum_{W \in \mathfrak{D}} \sum_{j=0}^{m(W)-1} f_*^j \rho_{|W}$$

is easily seen to be invariant by f and part 1 of Corollary 3.1 implies that it is finite.

We claim that the probability measure $\widetilde{\rho} := \widehat{\rho}(I)^{-1}\widehat{\rho}$ is an equilibrium state of f for the potential φ. To prove this, observe that by part 2 of Corollary 3.1 and [MU03, Theorem 2.2.9], the measure ρ is an equilibrium state of F for the potential Φ^φ. Therefore we have

$$h_\rho(F) + \int \Phi^\varphi \, d\rho = P(F, \Phi^\varphi) = 0.$$

By the generalized Abramov's formula [Zwe05, Theorem 5.1], we have $h_\rho(F) = h_{\widetilde{\rho}}(f)\widehat{\rho}(I)$, and by definition of $\widehat{\rho}$ we have $\int m \, d\rho = \widehat{\rho}(I)$. We thus obtain,

$$\begin{aligned} h_{\widetilde{\rho}}(f) = \widehat{\rho}(I)^{-1} h_\rho(F) = -\widehat{\rho}(I)^{-1} \int \Phi^\varphi \, d\rho \\ = -\widehat{\rho}(I)^{-1} \int \varphi \, d\widehat{\rho} + P(f, \varphi) = - \int \varphi \, d\widetilde{\rho} + P(f, \varphi). \end{aligned}$$

This shows that $\widetilde{\rho}$ is an equilibrium state of f for the potential φ.

To show that $\widetilde{\rho}$ is the unique equilibrium state of f for the potential φ, it is enough to show that the Lyapunov exponent of each equilibrium state of f for the potential φ is strictly positive, see [Dob12] for a proof of this result in the case of complex rational maps that extends to the interval maps considered here. In fact, if ν is such a measure, then

$$h_\nu(f) = P(f, \varphi) - \int \varphi \, d\nu \geq P(f, \varphi) - \sup_{J(f)} \varphi > 0.$$

By Ruelle's inequality, the Lyapunov exponent of ν is strictly positive. This proves that $\widetilde{\rho}$ is the unique equilibrium state of f for the potential φ. The argument above also shows that $h_{\widetilde{\rho}}(f) > 0$.

In view of part 1 of Corollary 3.1, to prove that $\widetilde{\rho}$ is exponentially mixing for f it is enough to apply Young's results in [You99] to the first return map of F to $V^{\widetilde{c}}$, as in the proof of [PRL07, Theorems B and C], or [PRL11, Theorem A].

It remains to show that the function $t \mapsto P(f, \varphi + t\psi)$ is real analytic on a neighborhood of $t = 0$. By Proposition B, the function $\mathscr{P}$ is finite and real analytic on a neighborhood of $(t, p) = (0, P(f, \varphi))$. On the other hand, by part 2 of Corollary 3.1 and [MU03, Proposition 2.6.13], we have

$$\frac{\partial}{\partial p} \mathscr{P}|_{(0, P(f, \varphi))} = - \int m(\cdot) \, d\rho < 0.$$

In view of Proposition B, applying the implicit function theorem to $\mathscr{P}$ at $(t, p) = (0, P(f, \varphi))$, we obtain that the function $t \mapsto P(f, \varphi + t\psi)$ is

real analytic on a neighborhood of $t = 0$. The proof of the Main Theorem is thus complete.

Appendix A. Hyperbolic potentials and the bounded range condition

Let X be a compact metric space, and let $T : X \to X$ be a continuous map. Recall that 2 continuous functions $\varphi : X \to \mathbb{R}$ and $\widetilde{\varphi} : X \to \mathbb{R}$ are *co-homologous*, if there is a continuous function $\chi : X \to \mathbb{R}$ such that

$$\widetilde{\varphi} = \varphi + \chi - \chi \circ T.$$

It is easy to see that every continuous potential $\varphi : X \to \mathbb{R}$ satisfying

$$\sup_X \varphi - \inf_X \varphi < h_{\mathrm{top}}(T), \tag{A.1}$$

also satisfies

$$\sup_X \varphi < P(T, \varphi); \tag{A.2}$$

so in the case f is as in the Main Theorem and $T = f|_{J(f)}$, such a potential φ satisfies hypothesis (1.3) of the Main Theorem.

The purpose of this section is to show that, under a fairly general condition on T, there is a potential φ satisfying (A.2) that is not co-homologous to any potential $\widetilde{\varphi}$ satisfying (A.1), with φ replaced by $\widetilde{\varphi}$. When f is a map in $\mathscr{A}$ that is topologically exact on $J(f)$, this general condition is easily seen to be satisfied when $T = f|_{J(f)}$.

Lemma A.1. *Let X be a compact metric space, let $T : X \to X$ be a continuous map, and let $h > 0$ be given. Suppose there are disjoint compact subsets X' and X'' of X that are forward invariant by T, and such that $h_{\mathrm{top}}(T|_{X'}) > 0$. Let $\varphi : X \to (-\infty, 0]$ be a continuous function that is constant equal to 0 on X', and such that $\sup_{X''} \varphi < -h$. Then*

$$\sup_X \varphi < P(T, \varphi),$$

and for every continuous function $\widetilde{\varphi} : X \to \mathbb{R}$ that is co-homologous to φ, we have

$$\sup_X \widetilde{\varphi} - \inf_X \widetilde{\varphi} > h. \tag{A.3}$$

Proof. By the variational principle there is a probability measure ν' on X that is supported on X', that is invariant by T, and such that $h_{\nu'}(T) = h_{\nu'}(T|_{X'}) > 0$. Then we have

$$P(T, \varphi) \ge h_{\nu'}(T) + \int \varphi \, d\nu' = h_{\nu'}(T) > 0 = \sup_X \varphi.$$

On the other hand, if $\widetilde{\varphi} : X \to \mathbb{R}$ is a continuous function that is co-homologous to φ, then we have

$$\sup_X \widetilde{\varphi} \ge \int \widetilde{\varphi} \, d\nu' = \int \varphi \, d\nu' = 0.$$

Moreover, if ν'' is a probability measure on X'' that is invariant by T, then

$$\inf_X \widetilde{\varphi} \leq \int \widetilde{\varphi}\, d\nu'' = \int \varphi\, d\nu'' = \int_{X''} \varphi\, d\nu'' \leq \sup_{X''} \varphi < -h.$$

Together with the inequality $\sup_X \widetilde{\varphi} \geq 0$ shown above, this implies (A.3). $\square$

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